CONJUGATE LINE CONGRUENCES CONTAINED IN A BUNDLE OF QUADRIC SURFACES*

BY

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Introduction.

The cubic complex defined by the generators of both systems in a bundle of quadric surfaces has been investigated by MONTESANO,† and his methods employed by FANO ‡ to determine a certain congruence of order three. A method, similar in part, had been previously found by STAHL, § in connection with congruences of the second order whose lines can be arranged on a system of quadric surfaces.

The questions discussed in these papers suggest the more general one of conjugate congruences,|| that is, those in which the two systems of generators of ∞^1 quadrics belonging to the bundle define two distinct congruences having the same focal surface. In the following paper I propose to derive such congruences and obtain a number of their properties. It will be seen that *the equations of such congruences of every order and of genus ranging from zero to a definite function of the order can be expressed rationally in terms of the eight points common to all the quadrics of the bundle.* The focal surface of each congruence remains invariant under a discontinuous birational transformation group of infinite order.

The preceding memoirs are all synthetic in their treatment; the present discussion is analytic, and the proofs of a few theorems already known will be given from a different point of view. Incidentally, extensive families of contact curves of a given quartic are obtained by algebraic methods.

* Presented to the Society, April 30, 1910.

† *Su di un complesso di rette di terzo grado*, Memorie di Bologna, ser. 5, vol. 3 (1893), pp. 549–577.

‡ *Nuove ricerche sulle congruenze di rette del 3° ordine*, Memorie di Torino, ser. 2, vol. 51 (1901), pp. 1–79.

§ *Ueber Strahlensysteme zweiter Ordnung*, Crelle's Journal, vol. 95 (1883), pp. 297–316.

|| The idea of conjugate congruences has been developed from a different point of view by BALDUS: *Ueber Strahlensysteme, welche unendlich viele Regelflächen 2 Grades enthalten*, Erlangen dissertation, 1910. The only case to which his method and my own both apply is that of congruences of order two.

§ 1. *The involution I.*

1. Let $H_1(x)=0$, $H_2=0$, $H_3=0$ be the equations of three quadric surfaces, and let $B_i(i=1,\cdots,8)$ be the eight points common to them. The bundle of quadrics

$$\Sigma=\lambda_1H_1+\lambda_2H_2+\lambda_3H_3=0 \tag{1}$$

defines a complex of order 3, the cone belonging to any point P being determined by the space quartic curve c_4 passing through P. This c_4 is the basis curve of the pencil of quadrics of (1) determined by P. The lines of the complex Σ can be put into (1, 1) correspondence with the points of space. Given any point $P\equiv(\xi_1,\xi_2,\xi_3,\xi_4)$, the line l joining P to B_1 will uniquely determine a quadric of Σ containing l. A generator m' of the other regulus passes through P. With P we may therefore associate the line m' of the complex. Conversely, any line m' of Σ will uniquely determine a quadric of the bundle. The quadric passes through B_1, through which passes a line l of the regulus to which m' does not belong. If l cuts m' in P, we may associate m' with the point P. When m' describes the whole regulus H determined by it, P will describe the line l; hence associated with every regulus is a line of the bundle B_1. The tangent plane to H at B_1 contains l and a line l' of the other regulus, so that there is associated with each quadric of Σ a pair of lines l, l' in the bundle B_1, each of which determines the other. These pairs determine a two-dimensional involution I.

The quadrics of Σ *are in* $(1,1)$ *correspondence with the pairs of lines of the involution* I *in the bundle* B_1.

2. Moreover, the quadrics of Σ are in $(1,1)$ correspondence with the points of the λ-plane. The pencil of quadrics in Σ passing through the point $P\equiv(\xi)$ corresponds to the straight line

$$\lambda_1H_1(\xi)+\lambda_2H_2(\xi)+\lambda_3H_3(\xi)=0.$$

The basis c_4 of the pencil of quadrics passes through B_1, hence the lines l, l' corresponding to quadrics of this pencil will describe a cubic cone K_3 passing through the other basis points B_i.

To the ∞^2 *straight lines of the* λ*-plane correspond the* ∞^2 *cubic cones containing the seven fixed lines* l_{1k} $(k=2,\cdots,8)$.

Any two lines in λ intersect in a point; two cones K_3 intersect in two lines which are conjugate in I; this pair of lines is the image of the given point.

The relations between (λ) and (H_2) are obtained by solving for $\lambda_1:\lambda_2:\lambda_3$ the equations of a quadric H and its tangent plane at B_1. Let $B_1\equiv(\eta_1,\eta_2,\eta_3,\eta_4)$, and write $\eta_ix_4-\eta_4x_i=y_i$. The equation of the tangent plane becomes $\Sigma\lambda_iv_i=0$, and H is of the form $\Sigma\lambda_iV_i+(\Sigma\lambda_iv_i)v_4=0$, wherein v_i, V_i are certain homogeneous linear and quadratic functions respectively in y_1,y_2,y_3, and $v_4=0$ defines a plane not passing through B_1. Solving for λ_i we have

$$(2) \qquad \lambda_1 : \lambda_2 : \lambda_3 = [v_2 V_3 - v_3 V_2) : (v_3 V_1 - v_1 V_3) : (v_1 V_2 - v_2 V_1).$$

3. Any congruence contained in Σ can be defined by a relation among the λ_i, or by a curve $f_n(\lambda)$ in the λ-plane. Corresponding to any line m of this congruence is a point P, the locus of which is the image cone of the given curve f_n. It will be of order $3n$ and have each line l_{1k} for an n-fold line. The order of the congruence will be $2n$ and the two systems of reguli will in general be contained in it.

§ 2. *Conjugate Congruences.*

4. It may happen that l, l' describe two different cones. For this purpose it is necessary that K_{3n}, image of an irreducible $f_n(\lambda)$, breaks up into two cones which are transformed into each other by the involution I. Such particular curves can be obtained by starting with the cone. Let l describe the cone $K_n(v)$ with vertex at B_1. From the relations

$$(3) \qquad \Sigma v_i \lambda_i = 0, \qquad \Sigma V_i \lambda_i = 0, \qquad K_n(v_1, v_2, v_3) = 0$$

the v_i should be eliminated. The resultant in λ is the equation of the required curve. If K_n passes through B_i, all the equations will be satisfied by its coördinates. If l_{ik} is the multiplicity of the line joining B_i to B_k, then Σl_{1k} must be subtracted from the order of the resultant which is $n + 2n$.

The image of a cone of order n passing l_{1k} times through B_k is a curve in the λ-plane of order $3n - \Sigma l_{1k}$. The lines of the cone and the points of the curve are in $(1, 1)$ correspondence. When l describes K_n, l' will describe another cone, birationally equivalent to it, and conjugate in the involution I. If the values of λ_i from (2) be substituted in the equation of the curve, K_n will be a factor. The other factor will give the equation of the other cone.

But the relations between l, l' can be determined directly. In the net of cubic cones $\Sigma a_i K_3^{(i)}$ a pencil is determined by one more basis line r. The cones of this pencil now contain eight fixed basis lines, hence they contain a ninth r'. If r_i be the coördinates of r, the relations between r, r' may be expressed by the equations

$$\frac{K_3^{(1)}(r')}{K_3^{(1)}(r)} = \frac{K_3^{(2)}(r')}{K_3^{(2)}(r)} = \frac{K_3^{(3)}(r')}{K_3\ (r)}.$$

When r describes a plane, r' will describe a rational cone of order 8, having the seven lines l_{1k} for three-fold edges. The involution I represents a Cremona transformation of order 8. If l describes a plane pencil passing through B_i, K' is of order 5 and has each of the remaining lines l_{1k} for double edge. If the plane passes through B_i, B_k, K' is a quadric cone through the other five.*

*This interesting transformation was first discussed by GEISER: *Ueber zwei geometrische Probleme*, Crelle's Journal, vol. 67 (1867), pp. 78–95; it has been further considered by BERTINI: *Ricerche sulle trasformazioni univoche involutorie nel piano*, Annali di Mate-

From the method in which it is produced it is evident that the Jacobian J_6 of the net, a sextic cone having each line l_{1k} for double edge, is invariant in I in such a way that every line will go into itself.

Every line in which $K(l)$ intersects J_6 must lie on $K'(l')$ also.

Since every line of J_6 goes into itself when operated upon by I, the corresponding quadric of Σ is a cone; J_6 is the projection from B_1 of the locus of the vertices of the cones contained in the bundle (1). Moreover, since the lines which K, K' have in common apart from those on J_6 occur in pairs, the image in the λ-plane is a double point on the image $f(\lambda)$ and the quadric H belongs to both the congruences σ, the locus of l, and σ', the locus of l'.

5. The two congruences σ, σ' have the same focal surface ϕ. Its order is twice the class of $f(\lambda)$. The points B_k are all singular upon ϕ, of an order half the order of the surface. The other isolated singular points are of order two; they are the vertices of the quadric cones belonging to the congruence and contained in Σ. Their number is the number of lines common to K and J_6 apart from the lines l_{1k}. The focal surface also contains a cuspidal curve and a double curve, but the points of these are not singular points for either congruence.

§ 3. *Double λ-Plane.*

6. A cone K whose points are the images of the lines of σ is a singular cone of σ', and conversely. It was seen that a $(1, 1)$ correspondence exists between the points of the λ-plane and the quadrics of the bundle Σ. Now consider the λ-plane as double, or two-sheeted, the same point defining the l or the l' generation of the image quadric according as it is regarded as belonging to the first or second sheet. For values of λ_i corresponding to a cone of Σ the two systems l, l' coincide, hence at each such point of the λ-plane the two sheets must coincide. The values of λ_i that make the discriminant $\Delta_4(\lambda)$ of Σ vanish define the cones. This is a general non-singular quartic curve in the λ-plane. When a locus $f(\lambda)$ crosses $\Delta(\lambda)$ the two sheets cannot be distinguished from each other, hence a necessary condition that $f(\lambda)$ may define two conjugate congruences is that it shall touch $\Delta(\lambda)$ at every point which they have in common.

In case $f(\lambda)$ is rational, this is also sufficient, since rational curves are unipartite. In such cases λ_i can be expressed rationally in terms of a parameter t, which when substituted in Σ will define a family of quadrics of the form

$$\sum_{m=1}^{n} t^m H_m = 0.$$

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matica, ser. 2, vol. 8 (1877), pp. 244–282. It has also received minor mention by CAPORALI in the Rendiconti di Napoli (1879), by DOEHLEMANN in Schlömilch's Zeitschrift, vol. 36 (1891), and by STEINMETZ in the American Journal of Mathematics, vol. 14 (1891), pp. 39–66.

The necessary and sufficient condition that a relation of genus zero between the coefficients λ_i *of the quadrics of a bundle may define two conjugate congruences is that the discriminant, expressed in terms of the parameter, shall be a square.*

For curves of genus greater than zero the condition is not sufficient on account of the existence of multipartite curves; one part may touch $\Delta(\lambda)$ in one region and another in another. However, the elimination of v_i from (3) will always define a contact curve of the proper kind. All the contact curves of $\Delta_4(\lambda)$ which belong to an odd characteristic can be obtained in this way.*

7. The equation of the focal surface is obtained by expressing the condition that the line $\Sigma H_i \lambda_i = 0$ shall touch the curve $f(\lambda) = 0$. If in the result $\phi(H_1, H_2, H_3) = 0$ we think of H_i as point coördinates, every quadric of the congruence is represented by a tangent to the curve ϕ, and the quartic curve along which the quadric touches the focal surface is the image of the point of contact. The residual points of intersection of the tangent and the curve have other quartic curves for images, hence:

Every quadric of a congruence contained in a linear bundle of quadrics touches the focal surface along a quartic curve and cuts it in a residual curve composed of space quartics of the first kind, all passing through the eight basis points of the bundle. Each of these residual quartics is itself the curve of contact of some quadric of the congruence.

Moreover, corresponding to every point of inflexion on $f(\lambda)$ there is a cusp on $\phi(H)$ and to every double tangent on f a double point on ϕ.

The focal surface has a cuspidal curve consisting of as many quartics as there are points of inflexion on $f(\lambda)$, *and a double curve consisting of as many quartics as there are bitangents of* $f(\lambda)$.

§ 4. *Problem of Projectivity.*

8. The points of a singular cone K with vertex at B_1 uniquely define a congruence σ. The cone K' is found from K by the involution I, hence the conjugate congruence σ' is also defined. But each point B_i is the vertex of a singular cone, and any one of these would be sufficient to define the same congruence. These cones are thus all birationally equivalent to the given cone, and each must satisfy the relations analogous to (2) and (3). The double points on ϕ which are the images of lines of J_6 do not need to be considered, for they are vertices of quadric cones belonging to both σ and σ'.

Any two bundles B_i, B_k can be put into $(1, 1)$ correspondence. A line m_i of B_i determines a quadric H of Σ. Since H passes through B_k there is just one line m_k through B_k lying on H and belonging to the same regulus as m_i.

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*For the literature of contact curves of $\Delta_4(\lambda)$ see the Encyklopädie der mathematischen Wissenschaften, III C5, §§ 60–62, 67, 74.

The quadric H passes through the other basis points $B_1, \cdots, B_8$, through each of which passes one line belonging to the other regulus, hence:

The six planes containing any line of the bundle B_i and any six basis points respectively are projective with the six planes through the corresponding lines of the remaining bundle and the same six basis points.

Since the eighth basis-point is uniquely fixed when the other seven are known, the problem of finding m_k when m_i is given is reduced to the problem of projectivity which, stated for the plane, is as follows: Given any five points $A_1, \cdots, A_5$, any other set of five points $A'_1, \cdots, A'_5$, and a point P, all in the same plane, to find a point P' such that the pencil $P'A'_i$ is projective with the pencil of five lines PA_i. This problem has been completely solved.*

The bundles B_i, B_k are equivalent under a Cremona transformation of order 5, having the remaining basis points for fundamental elements of the second order; to an arbitrary plane through B_i corresponds a rational quintic cone with vertex at B_k and having double edges passing through the other basis points. If the plane of B_i passes through B_1, the K_5 of B_1 is composite, consisting of a quadric cone through the other five edges and a cubic cone with a double edge passing through B_1 and passing once through each of the remaining basis points. If the pencil B_i passes through B_k, the K_3 of B_k passes through B_i. If the pencil B_i passes through B_1 and B_2, the proper image in B_k is the plane $B_kB_1B_2$. If K_n of B_i has a double or cuspidal edge not passing through any other basis point, its image in B_k will also have a double or cuspidal edge not passing through any other basis point, and similarly for its image in the involution I. The equations of this transformation can be obtained readily by three quadric inversions and two collineations, hence the complete system of singular cones of both σ and σ' can be expressed analytically when any one such cone of either is given.

§ 5. *Infinite Birational Groups.*

9. The focal surface ϕ is invariant under the operation of interchanging the points of contact of every line of σ. This transformation T_1 is birational and is of order 2. The congruence σ' defines another birational transformation T_2 of order 2. Except for particular cases arising from relations among the basis point B_i, the operation $T_1 T_2$ is of infinite order.†

The equations of this transformation can be obtained directly by the same method as that previously employed in the case of rational conjugate congruences.‡

*STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 1 (1908), pp. 352–358. See also SCHOENFLIES: *Encyklopädie der math. Wissenschaften*, III AB 5, § 16 (p. 442).

† SNYDER: *Infinite discontinuous groups of birational transformations which leave certain surfaces invariant*, these Transactions, vol. 11 (1910), pp. 15-24.

‡ SNYDER: *Surfaces invariant under infinite discontinuous birational groups defined by line congruences*, American Journal of Mathematics, vol. 32 (1910), pp. 177–185.

The fundamental points are the eight basis points and the vertices of the quadric cones belonging to both congruences; the fundamental curves are the curves of contact of these cones with the focal surface.

Among the quadrics of the congruence there are a finite number which touch the focal surface along two quartic curves. These double quadrics may arise in two different ways, and have in consequence different relations between the two quartic curves of contact.

If $K(v)$ has a double edge not passing through a basis point, $f(\lambda)$ will have a double point and $\phi(H)$ a corresponding double tangent. Let the images of the points of contact be c_4, c_4'. A line of σ will cut c_4 in P_1, P_2 and cut c_4' in P_1', P_2'. Corresponding to one sheet of $K(v)$ the first pair of points are foci of the given line, and to the other sheet, the second pair. Similarly, P_1 is focus for the σ' line passing through it, for the same sheet of $K(v)$.

Now suppose $K(v)$ intersects $K'(v)$ in a pair of conjugate lines l, l' in I. These two lines belong to the same quadric H, define a double point on $f(\lambda)$, and a double tangent to $\phi(H)$. Every line of both reguli of H belongs to both congruences σ and σ'. Let c_4, c_4' be the quartics of contact, and P a point on c_4. The lines m, m' both touch ϕ at P. As lines of σ, m has its foci on c_4, but m' on c_4'; as lines of σ', m has its foci on c_4', and m' on c_4. Every rational congruence of order > 2 contained in Σ must have a double quadric of one or the other of these forms. For orders higher than 3 both forms may appear in the same congruence.

The details of a few illustrative cases will now be discussed.

§ 6. *Images of Contact Curves.*

10. Let K_1 be a plane pencil with vertex at B_1 and passing through B_2, B_3. The image in λ is a bitangent to $\Delta_4(\lambda)$, hence the congruence is of order 1. Since $B_k (k = 4, 5, 6, 7, 8)$ is also the vertex of a pencil passing through B_2, B_3 it follows that the line B_2B_3 belongs to the σ' congruence, and all the lines of σ cut it. The pencil of quadrics in Σ which contains B_2B_3 has for its remaining basis curve a space cubic passing through the other six basis points and cutting B_2B_3 twice. The σ congruence therefore consists of the common secants of B_2B_3 and the associated cubic.

The images of the 28 bitangents of $\Delta_4(\lambda)$ *are the 28 lines* B_iB_k *and the 28 associated cubics passing through the remaining basis points.*

11. If K_1 passes through B_2 but not B_3, the λ image is a conic touching $\Delta_4(\lambda)$ in four points. The congruence is of order 2 and ϕ is of order 4. This is the well-known congruence of order 2, class 6, which has a singular plane. The plane K_1 can pass through any of the $8\cdot 7/1\cdot 2$ lines B_iB_k, and have ∞^1 positions in each case; each defines a contact conic of $\Delta_4(\lambda)$, hence:

The set of 28 series of ∞^1 *contact conics of* $\Delta_4(\lambda)$ *are images of* (*2*, *6*) *congruences having a singular plane.*

12. A K_1 with vertex at B_1 but not containing any other basis points will cut J_6 in six lines. The image K_8' in I will cut K_1 in these six lines and also in one pair l, l' conjugate in I. The λ-image is a nodal cubic touching $\Delta_4(\lambda)$ in six points. The congruence is rational, of order 3, class 9, and has one double quadric of the second kind. The focal surface is of order 8, has eight fourfold points and six double points. Its cuspidal curve is composed of three quartics passing through the eight fourfold points. The singular cones at the other basis points are rational K_5 having the lines joining all the others except B_1 for double edges. The conjugate congruence has B_1 for the vertex of a rational K_8 having triple edges passing through all the other basis points, and a K_4 from each of the remaining basis points with a triple edge passing through B_1 and passing simply through the remaining six. The vertices of the six quadric cones all lie in the given K_1.

13. A K_2 through five basis points will define the (1, 3) congruence already discussed. In fact the singular cones through B_2 and B_3 are exactly of this type.

A K_2 through four basis points defines the well known (2, 6) congruence without a basis plane.

The conjugate congruence is of the same form, hence:

The set of 35 series of ∞^1 *contact conics of* $\Delta_4(\lambda)$ *are images of* (*2*, *6*) *congruences having no singular plane.*

If K_2 has its vertex at B_1 and passes through B_2, B_3, B_4, it will cut J_6 in six lines apart from lines l_{ik}. The conjugate K_7' will have l_{12}, l_{13}, l_{14} for double edges, and the remaining l_{ik} for triple edges. It will therefore cut K_2 in one pair of lines l, l', conjugate in I. The image in the λ plane is a nodal cubic touching $\Delta_4(\lambda)$ in six points. The congruence is of order 3, class 9, has a double quadric of the second kind, and contains six quadric cones. The focal surface is of order 8 and has many properties in common with the preceding case, but the vertices of the quadric cones do not lie in a plane, and the configuration of the singular cones at the basis points is different.

The points B_2, B_3, B_4 are vertices of cones of order 6, K_6 having the lines to the remaining two for triple edges and double edges passing through the four points B_5, B_6, B_7, B_8. Each passes simply through B_1. The conjugate congruence has each of these three points for vertex of a cubic cone having a double edge, passing simply through those four basis points, but not through the two remaining ones of the triad. The given congruence has B_5, B_6, B_7, B_8 as vertices of singular cones of order four; each has a double edge through B_2, B_3, B_4, and passes simply through the remaining points of the tetrad, but does not pass through B_1. The conjugate congruence has each of the points B_5,

B_6, B_7, B_8 for vertex of a quintic cone having the line to B_1 for triple edge; the remaining points of the tetrad are on double edges, and each cone passes simply through the triad B_2, B_3, B_4. It is now easy to obtain congruences of orders 4, 5, 6 and having three, six, or ten double quadrics of the second kind as images of a quadric cone having two, one, or no lines l_{1k}. No rational congruence of order higher than six can have a singular cone of order less than three.

14. A K_3 with six simple lines l_{1k} will define a congruence of order 3 without a double quadric. If B_2 is the basis point not lying on $K_3(B_1)$, then B_2 is the vertex of a similar cone, passing through the same six basis points, which are vertices of K_5, each having lines joining the other five for double edges. The conjugate congruence has sextic cones at B_1 and B_2, each having the line B_1B_2 for triple edge and each having double edges through the other six basis points. The points $B_3, \cdots, B_8$ are vertices of K_4's having double edges passing through B_1 and B_2, and passing simply through the other basis points.

These two conjugate congruences are the only ones without double quadrics that can be mapped upon elliptic cones. Since there are 28 pairs B_iB_k and each cone contains three undefined constants, these congruences are imaged in the λ-plane by 28 series of ∞^3 contact cubics of $\Delta_4(\lambda)$. The remaining 36 series do not define conjugate congruences. If the original $K_3(B_1)$ has a double edge not containing any other basis point, the resulting rational congruences will have a double quadric of the first kind. *We have thus found six rational congruences of order 3 contained in* Σ; these results are believed to be new.*

15. Another interesting illustration is furnished by the $K_6(B_1)$ having double edges passing through all the other basis points. The singular cones at the other basis points are of the same kind, and the conjugate congruence has an exactly similar configuration. The image in λ is a non-singular quartic curve touching $\Delta_4(\lambda)$ in eight points. The congruence is therefore of order four. *The focal surface is of order 24, has eight twelve-fold points and eight double points; it has a cuspidal curve of order 96 composed of 24 quartics of the first kind passing through the eight singular points, and a nodal curve of order 112 composed of 28 quartics of the same bundle.* As before, K_6 may have one or more double edges, giving rise to quartic congruences having one or more double quadrics of the first kind.

The two congruences of order 2, of genus zero, the two conjugate congruences of order 3, of genus one, and the self-conjugate congruence of order 4, of genus three, are the only possible conjugate congruences contained in Σ which do not have a double quadric.

Congruences which can be mapped birationally upon cones of given genus can

* The elliptic congruences of order three were found by FANO (l. c.); he does not mention congruences having a double quadric.

be constructed, but the corresponding order is not less than the minimum order of a plane curve of that genus.

16. Among the $K_6(B_1)$ having seven double edges through the other basis points are two exceptions, J_6 and the image of a general c_2 in the λ-plane. The former corresponds to $\Delta_4(\lambda)$ itself and is defined as the congruence formed by the cones in Σ. From the preceding theorem we see that its focal surface is of order 24, has 24 cuspidal quartics and 28 double quartics, but from No. 10 the latter consists of 28 lines l_{ik} and the 28 associated cubics; moreover, instead of having eight other double points, it has as double point every point of the sextic which is projected from B_1 by J_6. The other exceptional type of K_6 is defined by quadratic relations among the cubics forming the net. Every such cubic is an adjoint of the sextic and every pencil of such cubics having one basis line of K_6 will, from the nature of its equation, have the other on K_6 also, that is, K_6 is hyperelliptic.*

The $c_2(\lambda)$ cuts $\Delta_4(\lambda)$ in eight points, but the vertices of the corresponding cones are not singular points on ϕ_4. If U_i be linear functions of the H_k, the equation has the form

$$U_1 U_3 - U_2^2 = 0.$$

When the equation of a quartic surface can be written as a quadratic function of three quadrics, the tangent cone of order 6 from each associated double point is either composite or hyperelliptic.

This theorem is true for surfaces of any order defined by functions of U_1, U_2, U_3.

§ 7. *Particular Positions of Basis Points.*

17. Let it be supposed that four basis points B_1, B_2, B_3, B_4 lie in a plane π_1, and the other four in another plane π_1'. When P describes a plane pencil with vertex at B_1, its image in the involution I will be a cone K_7' of order 7, with l_{12}, l_{13}, l_{14} for double edges, and the lines from B_1 to the other basis points for triple lines. As the basis points appear unsymmetrically a singular cone of order n and vertex at B_k will be designated by $K_n(k\,; l_h, m_p, \cdots)$, $l_h \cdots$ indicating that the line joining B_k to B_l is an h-fold line, m_p that $B_k B_m$ is a p-fold line, etc. Moreover, the involutions I will be different for different points. This difference will be indicated by subscripts. The result of operating with I_1 upon $K_1(1)$ will be indicated by the symbol

$$K_1(1) I_1 K_7'(1\,; 2_2, 3_2, 4_2, 5_3, 6_3, 7_3, 8_3).$$

The line $\pi_1 \pi_1'$ is now a part of the locus of vertices of quadric cones contained in the bundle Σ. The residual curve is a hyperelliptic quintic whose projecting cone from B_1 may be expressed by the symbol $J_5(1\,; 2_1, 3_1, 4_1, 5_2, 6_2, 7_2, 8_2)$.

* SNYDER: *On a special algebraic curve having a net of minimum adjoint curves*, Bulletin American Mathematical Society, vol. 14 (1907), pp. 70–74.

The discriminant curve $\Delta_4(\lambda)$ has one double point, the coördinates of which define the planes π_1, π_1' in Σ.

The 16 double tangents of $\Delta_4(\lambda)$ are images of the lines $B_k B_h$ ($k = 1, 2, 3, 4$; $h = 5, 6, 7, 8$) and the six tangents from the node are each the image of a pair of lines, one in π_1, one in π_1'. The transformation T_{ik}, defining the cone at B_k when that at B_i is given, is not changed from that of the general case when B_i, B_k both lie in π_1 or π_1' but is of order 4 when they lie in different planes. Thus we may write

$$K_1(1)\, T_{15} K_4(5\,; 2_2, 3_2, 4_2, 6_1, 7_1, 8_1).$$

The focal surface of the congruence of cones in Σ is now of order 20, contains 18 cuspidal quartics of the first kind, six double planes, six pairs of intersecting double lines, 16 other double lines, 16 double space cubics, and a double quintic.

The planes through B_1 and any basis point in π_1' define the general $(2, 5)$ congruences. The line of intersection of this plane and π_1 does not pass through any other basis point, hence it is the image of a cone breaking up into π_1 and π_1'. The cone consists of two plane pencils, one in π_1, the other in π_1', the vertices of each being on the line $\pi_1 \pi_1'$. In the conjugate congruence the planes of these pencils are interchanged.

The image conic in the λ-plane touches $\Delta_4(\lambda)$ in three points and passes through the node.

The singular cone $K_1(1;\ 2_1)$ defines a particular form of the (2, 6) congruence having one singular plane.

A general $K_1(1)$ defines a $(3, 8)$ congruence, in which one cone breaks up into π_1, π_1'. The focal surface has one more double point and two more double planes than in the general case. The congruence contains one double quadric of the second kind.

An interesting example is the $(3, 7)$ congruence defined by the singular cone $K_2(1;\ 5_1, 6_1, 7_1)$. The complete configuration of singular cones of the two conjugate congruences is as follows:

$$K_2(1;\ 5_1, 6_1, 7_1), \qquad K_4(2;\ 3_1, 4_1, 5_2, 6_2, 7_2, 8_1),$$
$$K_5'(1;\ 2_1, 3_1, 4_1, 5_2, 6_2, 7_2, 8_3), \qquad K_3'(2;\ 1_1, 5_1, 6_1, 7_1, 8_2),$$
$$K_4(5;\ 1_1, 2_2, 3_2, 4_2, 6_1, 7_1),$$
$$K_3'(5;\ 1_1, 2_1, 3_1, 4_1, 8_2).$$

Those at B_3, B_4 can be obtained from those at B_2 by interchanging 2 and 3, 2 and 4 respectively, and those at B_6, B_7, B_8 by making similar changes in those at B_5.

The congruences contain a double quadric of the second kind consisting of the two pencils in π_1, π_1'. The image in the λ-plane consists of a cubic having a double point at the node of $\Delta_4(\lambda)$ and touching the latter in four points.

The cone $K_2(1;\ 4_1,\ 5_1,\ 6_1)$ defines a $(3,\ 8)$ congruence having five cones, one pair of planes, and a double quadric of the second kind.

The cone $K_2(1;\ 3_1,\ 4_1,\ 5_1)$ defines a $(3,\ 9)$ congruence containing six cones of Σ and a double quadric of the second kind.

The cubic cone $K_3(1;\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_1,\ 8_1)$ defines a $(3, 8)$ congruence having 5 quadric cones of Σ. If K_3 is rational the congruence contains a double quadric of the first kind.

The cone $K_3(1;\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_1)$ defines a $(3,\ 9)$ congruence as in the general case.

The preceding eighteen forms are the only possible types of congruences in Σ *of order less than 4.*

The non-hyperelliptic cone of order 6 having a double edge passing through each remaining basis point defines a $(4,\ 12)$ congruence as in the general case. The conjugate congruence is of the same type.

18. Let the basis points lie on two pairs of planes:

$$\pi_1 \equiv B_1B_2B_3B_4, \qquad \pi_1' \equiv B_5B_6B_7B_8;$$
$$\pi_2 \equiv B_1B_2B_5B_6, \qquad \pi_2' \equiv B_3B_4B_7B_8.$$

The two lines $\pi_1\pi_1'$ and $\pi_2\pi_2'$ and a space quartic of the first kind constitute the locus of the vertices of cones in Σ. The latter is projected from B_1 by the cone $J_4(1;\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_2,\ 8_2)$. The involution I transforms a plane pencil into a cone of order 6 of the form

$$K_1(1)IK_6'(1;\ 2_1,\ 3_2,\ 4_2,\ 5_2,\ 6_2,\ 7_3,\ 8_3).$$

All the operations T_{ik} transform plane pencils into quartic cones except when ik is one of the combinations 12, 34, 56, or 78. Thus we have

$$K_1(1)T_{12}K_5(2;\ 3_2,\ 4_2,\ 5_2,\ 6_2,\ 7_2,\ 8_2); \quad K_1(1)T_{13}K_4(3;\ 2_2,\ 4_1,\ 5_2,\ 7_1,\ 8_1).$$

Of the congruences of the second order, $K_1(1;\ 2_1)$ defines a particular $(2,\ 6)$; $K_1(1;\ 3_1)$ defines a particular $(2,\ 5)$, and $K_1(1;\ 7_1)$ a general $(2,\ 4)$.

Of the congruences of order 3, $K_1(1)$ defines a $(3,\ 7)$ having four cones of Σ, two pairs of planes, and a double quadric of the second kind. The cone $K_2(1;\ 6_1,\ 7_1,\ 8_1)$ defines a $(3,\ 7)$ congruence having a pair of planes for a double quadric of the second kind. The image cubic curve in the λ-plane has a double point at one node of $\Delta_4(\lambda)$ and passes simply through the other; it touches $\Delta_4(\lambda)$ in three other points.

19. The basis points may lie in three pairs of planes:

$$\pi_1 \equiv B_1B_2B_3B_4, \quad \pi_1' \equiv B_5B_6B_7B_8, \quad \pi_2 \equiv B_1B_2B_5B_6, \quad \pi_2' \equiv B_3B_4B_7B_8,$$
$$\pi_3 \equiv B_1B_3B_6B_8, \quad \pi_3' \equiv B_2B_4B_5B_7.$$

The locus of the vertices of cones in Σ now consists of the three lines $\pi_1\pi_1'$, $\pi_2\pi_2'$, $\pi_3\pi_3'$, and a space cubic, the latter being projected from B_1 by the cone $J_3(1\,;\ 4_1,\ 5_1,\ 7_2,\ 8_2)$. The involution I transforms a plane pencil into a cone of order 5 of the form $K_1(1)IK_5'(1\,;\ 2_1,\ 3_1,\ 4_2,\ 5_2,\ 6_1,\ 7_3,\ 8_2)$. The transformations T_{ik} can be most easily determined from the details of the $(3,6)$ congruence defined by $K_1(1)$. The singular cones of both this congruence and its conjugate at B_1 are given above; the others are as follows:

$$
\begin{array}{ll}
K_4(2\,;\ 3_2,\ 4_1,\ 5_1,\ 6_2,\ 7_1,\ 8_2), & K_4(3\,;\ 2_2,\ 4_1,\ 5_2,\ 6_2,\ 7_1,\ 8_1),\\
K_2'(2\,;\ 1_1,\ 7_1,\ 8_1), & K_2'(3\,;\ 1_1,\ 7_1,\ 8_1),\\
K_3(4\,;\ 2_1,\ 3_1,\ 5_1,\ 6_2,\ 8_1), & K_4(6\,;\ 2_2,\ 3_2,\ 4_2,\ 5_1,\ 7_1,\ 8_1),\\
K_3'(4\,;\ 1_2,\ 5_1,\ 6_1,\ 7_1,\ 8_1), & K_2'(6\,;\ 1_1,\ 4_1,\ 7_1),\\
K_3(5\,;\ 2_1,\ 3_2,\ 4_1,\ 5_1,\ 8_1), & K_3(8\,;\ 2_2,\ 3_1,\ 4_1,\ 5_1,\ 6_1),\\
K_3'(5\,;\ 1_2,\ 3_1,\ 4_1,\ 7_1,\ 8_1), & K_3'(8\,;\ 1_2,\ 2_1,\ 4_1,\ 5_1,\ 7_1),
\end{array}
$$

$$K_2(7\,;\ 2_1,\ 3_1,\ 5_1),$$

$$K_4'(7\,;\ 1_3,\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 8_1).$$

The image of the congruence of cones is a trinodal quartic in the λ plane. Its four proper bitangents are images of the congruences cutting the lines l_{17}, l_{28}, l_{35}, l_{45} and their associated space cubics.

Of the congruences of order 2, $K_1(1\,;\ 7_1)$ defines the general $(2,3)$ case. Its image in the λ-plane is a conic passing through all three nodes of $\Delta_4(\lambda)$ and touching it once. The cone $K_1(1\,;\ 8_1)$ defines a particular $(2,4)$ congruence, and $K_1(1\,;\ 2_1)$ defines a particular $(2,5)$. A $(2,6)$ congruence cannot be contained in Σ in this case.

Among the congruences of order 3, that of lowest class is the $(3,5)$ defined by $K_2(1\,;\ 5_1,\ 7_1,\ 8_1)$. It contains four of the basis planes simply and the remaining pair doubly. The section of the focal surface made by the latter consists of two conic sections, each counted twice. The image in the λ-plane is a cubic having a double-point at one of the double-points of $\Delta_4(\lambda)$, passing simply through each of the remaining nodes, and touching $\Delta_4(\lambda)$ in two other points.

As in the preceding cases, the non-hyperelliptic $K_6(1,\ 2_2,\ \cdots,\ 8_2)$ defines a $(4,12)$ congruence with a singular cone of order 6 at each basis point, and containing 8 quadric cones of Σ.

20. The eight basis points may be arranged on four pairs of planes. In addition to the preceding pairs of planes π_i, π_i' we may write

$$\pi_4 \equiv B_1B_4B_5B_8, \qquad \pi_4' \equiv B_2B_3B_6B_7.$$

The locus of the vertices of cones now consists of the four lines $\pi_1\pi_1'$, etc., and

of their two transversals. The latter project from B_1 into $J_2(1\,;\ 7_2)$. The curve $\Delta_4(\lambda)$ consists of two conics passing through the images of the four lines $\pi_1\pi_1'$, etc. In the involution I,

$$K_1(1)IK_4'(1\,;\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_3,\ 8_1),$$

that is, in the Cremona transformation defined by the involution I, the image of a plane passing through B_1 is in this case a quartic cone having its vertex at B_1, containing B_1B_7 as triple edge and passing simply through the remaining basis points.

The only congruences of the second order that can exist in this case are a particular $(2,\ 4)$ defined by $K_1(1;\ 2_1)$ and the general $(2,\ 2)$ defined by $K_1(1\,;\ 7_1)$. In the latter all the basis-points are vertices of plane pencils, and the conjugate congruence has the same configuration of singular planes. The image in the λ-plane consists of a conic belonging to the pencil defined by the factors of $\Delta_4(\lambda)$. The focal surface is the general Kummer surface.

Of congruences of the third order, $K_1(1)$ defines a $(3,\ 5)$ having a double quadric of the second kind; the elliptic $K_3(1\,;\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_1)$ defines a $(3,\ 7)$ without a double quadric, and $K_3(1\,;\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 8_1)$ defines a particular form of $(3,\ 9)$. When the cone is rational the corresponding congruence acquires a double quadric of the first kind.

As in the general case, the non-hyperelliptic $K_6(1\,;\ 2_2,\ \cdots,\ 8_2)$ defines a $(4, 12)$ congruence without any singular planes.

21. The basis points may be arranged on five pairs of planes, the four preceding, and

$$\pi_5 \equiv B_1,\ B_2,\ B_7,\ B_8, \qquad \pi_5' \equiv B_3,\ B_4,\ B_5,\ B_6.$$

The locus of the vertices of cones consists of the five lines $\pi_1\pi_1'$ etc., and of one other line, the latter projecting from B_1 into $J_1(1:7_1)$. The curve $\Delta_4(\lambda)$ consists of a conic and a pair of straight lines, each of the latter defining a pencil of quadric cones having common vertices. In the involution I, the image of a plane through B_1 is a cubic cone having B_1B_7 for double edge and passing simply through B_3, B_4, B_5, B_6. The symbol becomes

$$K_1(1)IK_3'(1:\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_2).$$

There are three particular congruences of the second order, a $(2,\ 4)$ defined by $K_1(1:2_1)$, a $(2,\ 3)$ by $K_1(1:3_1)$, and a $(2,\ 2)$ by $K_1(1:7)$.

Besides the rational (3, 4) congruences defined by $K_1(1)$, the elliptic (3, 6) defined by $K_3(1:\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_1)$ is of interest. For the first time the conjugate congruence of that defined by $K_6(1\,;\ 2_2,\ \cdots,\ 8_2)$ is not directly expressed by substitution of the values of the coördinates. The lines l_{11}, l_{18} are already three fold in the extraneous factors that divide out of $K_8'(1:\ 2_3,\ \cdots,\ 8_3)$, hence

are neutral. But by substituting the coördinates of these lines in K_6, both appear as double edges.

22. The basis points may be arranged on six pairs of planes, the five preceding and also

$$\pi_6 \equiv B_1 B_4 B_6 B_7, \qquad \pi_6' \equiv B_2 B_3 B_5 B_8.$$

The locus of the vertices of cones consists of the six lines $\pi_i \pi_i'$, four of which meet in a point. The cones having this point for vertex contain four common edges each containing two basis points, hence form a pencil belonging to Σ. The curve $\Delta_4(\lambda)$ consists of the sides of a quadrilateral. The involution I is defined by the transformation

$$K_1(1) I K_2'(1:3_1, 5_1, 7_1).$$

The plane $K_1(1)$ defines a $(3, 3)$ congruence having a double quadric of the first kind. It contains no cones, but has six pairs of planes. The conjugate congruence is of the same type. Congruences of order 3 and class 2 are impossible when the basis points are distinct. Particular $(2, 2)$ and $(2, 3)$ forms as well as $(3, k)$, $k = 3, \cdots, 6$, can be obtained. The involution reduces to ordinary quadratic inversion. As in the preceding cases $K_6(1; 2_2, \cdots, 8_2)$ defines a congruence of type $(4, 12)$, having a conjugate of the same kind.

When the points are arranged on six pairs of planes, they may be assumed to be $(\pm 1, \pm 1, \pm 1, 1)$. The equation of the bundle of quadrics is of the form

$$\lambda_1(x_1^2 - x_4^2) + \lambda_2(x_2^2 - x_4^2) + \lambda_3(x_3^2 - x_4^2) = 0,$$

and $\Delta_4(\lambda)$ becomes

$$\lambda_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3) = 0.$$

The equation of a line through $B_1 \equiv (1, 1, 1, 1)$ are $x_i = 1 + \rho\xi_i$, from which, if it belongs to Σ_1,

$$\lambda_1(\xi_1 - \xi_4) + \lambda_2(\xi_2 - \xi_4) + \lambda_3(\xi_3 - \xi_4) = 0,$$

$$\lambda_1(\xi_1^2 - \xi_4^2) + \lambda_2(\xi_2^2 - \xi_4^2) + \lambda_3(\xi_3^2 - \xi_4^2) = 0.$$

A congruence of the second order is defined by $K_1(1;\ 2_1)$, wherein $B_2 \equiv (1, -1, 1, 1)$. This cone requires relations of the form

$$A(\xi_1 - \xi_4) = B(\xi_2 + \xi_4) = C(\xi_3 - \xi_4)$$

among the ξ_i. The ξ-eliminant of these equations is the equation of the λ-curve, image of the congruence; it is the conic

$$(c\lambda_1 + A\lambda_3)^2 + \lambda_2(c^2\lambda_1 + A^2\lambda_3) = 0,$$

which passes through the vertices $(0, 1, 0)$, $(0, -1, 1)$, $(1, -1, 0)$ of the quadrilateral and touches the line $\lambda_2 = 0$ at $(A, 0, -c)$. The $(2, 3)$ congru-

ence contains one cone of Σ besides four singular cones whose vertices are at the basis points, and three pairs of planes besides the four singular pencils at four basis points. The conjugate is of the same form.

23. Instead of a finite number of basis points, the bundle Σ may have a basis curve. There are four general forms.

(1) When H_1, H_2, H_3 can be written in the form

$$u_1 v_2 - u_2 v_1 = 0,\ u_1 w_2 - u_2 w_1 = 0,\ v_1 w_2 - v_1 w_1 = 0\,,$$

wherein u_i, v_i, w_i are linear in the variables x_i, the bundle has a space cubic for basis curve.

(2) The equation

$$x_4(\lambda_1 u_1 + \lambda_2 u_2) + \lambda_3(x_1 x_2 - x_3^2) = 0$$

defines a bundle of quadrics having the conics $x_4 = 0$, $x_1 x_2 - x_3^2 = 0$ for basis curve and the two basis points $u_1 = 0$, $u_2 = 0$, $x_1 x_2 - x_3^2 = 0$.

(3) The equation

$$\alpha x_1 x_4 + \beta x_1 x_3 + \gamma x_2 x_3 + \delta x_2 x_4 = 0\,,$$

in which α, β, γ, δ are linear homogeneous functions of λ_1, λ_2, λ_3, defines a bundle having the two basis lines $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, $x_4 = 0$.

(4) The bundle defined by

$$x_1 u_4 - x_4 u_1 = 0\,, \qquad x_2 u_4 - x_4 u_2 = 0\,, \qquad x_3 u_4 - x_4 u_3 = 0$$

has the line $x_4 = 0$, $u_4 = 0$ for basis curve. It has four basis points not lying on the line.

Any two quadrics contained in one of these forms (1), $\cdots$, (4) will intersect in the basis curve and a residual curve of order 1, 2, 2, or 3. In the first and third cases the residual intersection consists of one or of two generators. In the fourth case only will the residual intersection be a curve which is cut by every generator of one system in more than one point. On the focal surface of any congruence contained within the bundle the points of contact with a line of the congruence lie upon the complete intersection of two quadrics of the bundle. In the first two cases the lines are not proper bitangents; in the third the focal surface is a ruled surface contained within a linear congruence, while in the fourth the points of contact of lines of one congruence lie upon a bundle of space cubics of which they are bisecants. The operation of interchanging these points of contact is a birational transformation of order two. In case (3) it is simply an axial involution, hence case (4) will be the only one to be considered as defining a non-linear transformation.

24. The two systems of generators are rationally separable, hence the cubic complex is composite. One system may be written in the form

$$\frac{\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3}{u_4} = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3}{x_4} = \sigma\,,$$

and the other becomes

$$\frac{\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3}{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3} = \frac{u_4}{x_4} = \sigma';$$

thus it is seen that Σ consists of a quadratic and of a special linear complex, the axis of the latter being the basis line $u_4 = 0$, $x_4 = 0$.

The four distinct basis-points are the vertices of the invariant tetrahedron of the collineation defined by

$$x_i = \rho u_i.$$

A congruence of order n is defined by a relation of the form $f_n(\lambda_1, \lambda_2, \lambda_3) = 0$.

As before, if ν is the class of f_n, the order of the focal surface will be 2ν; the line $u_4 = 0$, $x_4 = 0$ is an ν-fold line upon the surface, and each of the distinct basis points is also ν-fold upon it. Upon the focal surface there is moreover a cuspidal curve composed of as many space cubics as $f_n(\lambda)$ has points of inflexion.

One or more of the quadric cones contained in σ may break up into a pair of planes; one plane must contain the basis line and pass through one basis point, and the other contains the three remaining basis points, hence the maximum number of such degenerate cones is four.

The equation

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = \sigma u_4$$

defines a pencil of planes through every tangent line to the curve $f_n(\lambda)$ in $u_4 = 0$. The equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = \sigma x_4$$

defines a similar pencil through the tangents to the curve $f_n(\lambda)$ in $x_4 = 0$. From the equations $x_i = \rho u_i$ these two systems are projective and the congruence is defined as the locus of intersection of corresponding planes. The complex σ is thus a tetrahedral complex.

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